SIMPLE AND DIRECT PROOF OF MACLANE's GRAPH PLANARITY CRITERION

Alexander Kelmans

Rutgers University, New Brunswick, New Jersey
University of Puerto Rico, San Juan, Puerto Rico

Abstract

We give a simple proof of MacLane's algebraic planarity criterion for graphs. This proof does not use any other known planarity criteria.

Keywords: graph, planarity, cycle space, a simple basis of a graph.

1 Introduction

We consider undirected graphs with no loops (parallel edges are possible). All notions on graphs, that are not defined here, can be found in [1, 12].

There are various graph planarity criteria. Here are some of them.

1.1 (Kuratowski [7]) *A graph is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$.*

1.2 (Whitney [13]) *A graph is planar if and only if it has a matroid dual graph.*

1.3 (MacLane [8]) *A graph is planar if and only if its cycle space has a 2–basis* (*i.e. a basis that consists of some cycles of the graph and such that every edge of the graph belongs to at most two cycles from the basis*).

A cycle C in a connected graph G is called *separating* if G/C has more blocks then G, and *non-separating*, otherwise.

1.4 (Kelmans [2, 3]) *A 3–connected graph is planar if and only if each edge of the graph belongs to exactly two non-separating cycles of the graph.*

There are several fairly simple proofs of **1.1** (e.g. [3, 9, 10]). Theorems **1.2** and **1.3** follow from **1.1** because K_5 and $K_{3,3}$ have no matroid dual graph and have no 2–basis, respectively (e.g. [1, 8, 13]). In [2, 3] we gave a simple proof of **1.4** that does not use any other known planarity criteria. We also gave a simple proof of **1.2** using **1.4**. Moreover, we showed that

1.5 [2] *A 3–connected graph has an edge belonging to at least three non-separating cycles if and only if it has a subdivision of K_5 or $K_{3,3}$.*

This fact implies that **1.1** follows from **1.4** and vise versa and that **1.3** follows from **1.4**.

The following theorem, due to W. Tutte [11] and, independently, A. Kelmans [2, 3], is an important result in the study of the graph cycle spaces.

1.6 *The set of non-separating circuits of a 3–connected graph generates the cycle space of the graph.*

In [2] we noted that **1.4** follows from **1.3** and **1.6**.

In this paper we give a simple proof of (a natural refinement of) MacLane's graph planarity criterion **1.3**. This proof does not use any other known planarity criteria.

More information on this topic (in particular, some strengthenings of **1.1**, **1.2**, and **1.4**) can be found in the expository paper [4] and in [5].

The results of this paper were presented at the Moscow Discrete Mathematics Seminar in 1977 (see also [6]).

2 Main notions and notation

Let G be a graph, $V(G)$ and $E = E(G)$ the sets of vertices and edges of G, respectively. Let $e(G) = |E(G)|$. If C is a cycle of G, then $E(C)$ is called a *circuit* of G. If $X, Y \subseteq E$, then let $X + Y$ denote the symmetric difference of X and Y, i.e. $X + Y = (X \cup Y) \setminus (X \cap Y)$. Then 2^E forms a vector space over $GF(2)$. Let $\mathcal{C}(G)$ denote the set of circuits of G, and so $\mathcal{C}(G) \subseteq 2^E$. Let $\mathcal{CS}(G)$ denote the subspace of 2^E generated by $\mathcal{C}(G)$. This subspace is called the *cycle space of* G. Obviously $X \in \mathcal{CS}(G)$ if and only if every vertex v in the subgraph of G, induced by X, has even degree. In particular, $\emptyset \in \mathcal{CS}(G)$. A basis B of $\mathcal{CS}(G)$ is called *simple* if every edge of G belongs to at most two members (edge sets) of B.

If $\mathcal{F} \subseteq 2^E$ and H is a subgraph of G, we write $H \in \mathcal{F}$ and $\mathcal{F} \setminus \{H\}$ instead of $E(H) \in \mathcal{F}$ and $\mathcal{F} \setminus \{E(H)\}$, respectively.

If $X \subseteq E(G)$, then let $\dot{X}$ denote the subgraph of G induced by X.

If H is a plane 2–connected graph, then let $\mathcal{F}(H)$ be the set of facial circuits of H.

A path P with end-vertices x and y is called a *path-chord of* a cycle C (and of the corresponding circuit $E(C)$) in G if $V(C) \cap V(P) = \{x, y\}$, and $E(C) \cap E(P) = \emptyset$.

A *thread* in G is a path T in G such that the degree of every inner vertex of T is equal to two and the degree of every end-vertex of T is not equal to two in G. Obviously, if C is a cycle of G and $E(C) \cap E(T) \neq \emptyset$, then $T \subseteq C$. If T is a thread in G, we write $G - (T)$ instead of $G - (T - End(T))$.

3 Proof of MacLane's planarity criterion

It is easy to see the following.

3.1 *Let G be a 2–connected graph and G not a cycle. Then G has a thread T such that $G - (T)$ is a 2–connected graph.*

Obviously

3.2 *Let G be a 2–connected planar graph, G_ϵ be an embedding of G into the plane, and F a facial circuit of G_ϵ. Then $\mathcal{F}(G) \setminus \{F\}$ is a simple basis of $\mathcal{CS}(G)$.*

3.3 *Let G be a 2–connected graph and G not a cycle. If B is a simple basis of $\mathcal{CS}(G)$, then G is planar and there is an embedding G_ϵ of G such that $B = \mathcal{F}(G) \setminus \{F\}$ for some $F \in \mathcal{F}(G)$.*

Proof We prove our claim by induction on $e(G)$. If $e(G) = 3$, then our claim is obviously true. So let $e(G) \geq 4$. By **3.1**, there is a thread T of G such that $G' = G - (T)$ is 2–connected. Since G is 2–connected, T belongs to a cycle of G. Therefore $E(T)$ belongs to at least one member of B. Since B is a simple basis of $\mathcal{CS}(G)$, $E(T)$ belongs to at most two members of B.

If $E(T)$ belongs to exactly one member of B, say C, then let $B' := B \setminus \{C\}$. If $E(T)$ belongs to (exactly) two members, say S and Z, of B, then let $B' := B \setminus \{S, Z\} \cup \{S + Z\}$. Then B' is a simple basis of G'.

By the induction hypothesis, G' is planar and there is an embedding G'_α of G' such that $B' = \mathcal{F}(G'_\alpha) \setminus \{D\}$ for some $D \in \mathcal{F}(G'_\alpha)$, and so every member of B' is a facial circuit of G'_α and every edge in $E(G) \setminus D$ belongs to exactly two facial circuits of G'_α that are members of B'.

Suppose that $B' = B \setminus \{C\}$. Since B is a simple basis of G and B' is a subset of B, clearly $C \setminus E(T)$ is a subset of D. Since T is a thread in G and C is an element of the cycle space of G, clearly $\dot{C} - (T)$ is a path, and so $\dot{C}$ is a cycle in G and T is a path-chord of cycle $\dot{D}$. Now since D is a facial circuit of G'_α, we can embed T in the face, bounded by $\dot{D}$, to obtain from G'_α an embedding G_ϵ of G, and so G is planar and $B = \mathcal{F}(G_\epsilon) \setminus \{C'\}$, where $\dot{C}'$ is the cycle in $\dot{D} \cup T$ containing T and distinct from $\dot{C}$.

Now suppose that $B' = B \setminus \{S, Z\} \cup \{S + Z\}$. Then $S + Z$ is a facial circuit of G'_α which is a member of B'. We know that $E(T) \subseteq S \cap Z$. Suppose that there is $e \in (S \cap Z) \setminus E(T)$. Then $e \in E(G') \setminus (S + Z)$, and so e belongs to a member, say R, of B'. Therefore e belongs to three members of B, namely, R, S, and Z, and so B is not a simple basis of $\mathcal{CS}(G)$, a contradiction. Thus $S \cap Z = E(T)$, and so $S + Z = (S \cup Z) \setminus E(T)$ and T is a path-chord of facial circuit $S + Z$ of G'_α. Then we can embed T in the face, bounded by $S + Z$, to obtain from G'_α an embedding G_ϵ of G, and so G is planar and $\mathcal{F}(G_\epsilon) = \mathcal{F}(G'_\alpha) \setminus \{S + Z\}) \cup \{S, Z\}$.

If $D = S + Z$ then $\mathcal{F}(G_\epsilon) = B$, and so the sum of members of B is equal to $\emptyset$. Therefore B is not a basis of $\mathcal{CS}(G)$, a contradiction. Thus $D \neq S + Z$, and so $B = \mathcal{F}(G_\epsilon) \setminus \{D\}$. □

Now we are ready to prove the following refinement of **1.3**.

3.4 *Let G be a 2–connected graph.*

(a) *The following are equivalent:*

$(a1)$ *G is planar and*

$(a2)$ *G has a simple cycle basis.*

(b) *Moreover, if G is not a cycle and S is a simple basis of $\mathcal{CS}(G)$ then there exists an embedding G_ϵ of G such that $S = \mathcal{F}(G_\epsilon) \setminus \{F\}$ for some $F \in \mathcal{F}(G_\epsilon)$.*

Proof By **3.2**, $(a1) \Rightarrow (a2)$. We prove (b) and $(a2) \Rightarrow (a1)$. If G is a cycle, our claim is obviously true. If G is not a cycle, then our claim follows from **3.3**. □

References

[1] R. Deistel, *Graph Theory*, Springer–Verlag, New York, 2000.

[2] A. Kelmans, The concept of a vertex in a matroid, the non-separating cycles, and a new criterion for graph planarity. In *Algebraic Methods in Graph Theory*, Vol. 1, Colloq. Math. Soc. János Bolyai, (Szeged, Hungary, 1978) North–Holland **25** (1981) 345–388.

[3] A. Kelmans, A new planarity criterion for 3–connected graphs, *J. Graph Theory* **5** (1981) 259–267.

[4] A. Kelmans, Graph planarity and related topics, *Contemporary Mathematics* **147** (1993), 635-667.

[5] A. Kelmans, On graph planarity and semi–duality, *Discrete Mathematics* **230**, 1–3, (2001) 149–166.

[6] A. Kelmans, Simple and direct proof of Maclane's graph planarity criterion, *RUTCOR Research Report 3–2005*, Rutgers University (2005).

[7] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* 15 (1930) 271–283.

[8] S. MacLane, A combinatorial condition for planar graphs, *Fund. Math.* 28 (1937) 22–32.

[9] Y. Makarychev, A short proof of Kuratowski's graph planarity criterion, *J. Graph Theory* **25** (1997) 129–131.

[10] C. Thomassen, Planarity and duality of inite and infinite graphs, *J. Combinatorial Theory* B–29 (1980) 244–271.

[11] W. Tutte, How to draw a graph, *Proc. London Math. Soc.* **13** (1963) 743-767.

[12] H.-J. Voss, *Cycles and Bridges in Graphs*, Deutscher Verlag der Wissenschaften, Berlin; Kluwer Academic Publisher, Dordreht, Boston, London, 1991.

[13] H. Whitney, Planar graphs, *Fund. Math.* **21** (1933) 73–84.